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1971 J. Phys. A: Gen. Phys. 4 231

(http://iopscience.iop.org/0022-3689/4/2/008)

J. Phys. A: Gen. Phys., 1971, Vol. 4. Printed in Great Britain

Approximate formulae for the superposition of coherent and chaotic fields

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MS. received 22*nd June* 1970, *in revised form* 25*th August* 1970

Abstract. Recently derived formulae for the integrated intensity distribution, the photon-counting distribution and its factorial moments in the statistics of the superposition of coherent and chaotic multimode fields are proposed as approximate formulae for light of arbitrary spectrum. It is shown by explicit calculations of the third factorial moment for the superposition of a one-mode coherent field with a Gaussian–Lorentzian field that the proposed formulae hold with good accuracy over a wide range of conditions. An application to the determination of spectral parameters of light is given.

1. Introduction

In 1959 Mandel obtained a formula for the photon-counting distribution as a generalization of the Bose–Einstein distribution to systems with more than one degree of freedom. Later it was pointed out by Bédard *et al.* (1967) that Mandel's formula can serve as an approximate formula for the photon-counting distribution of a chaotic field of arbitrary spectral density if a free parameter of the number of degrees of freedom is adjusted in such a way that the second moments of the number of counts for the exact formula and the approximate formula coincide. Of course, the exact formula cannot be obtained in a close form. They have shown by explicit calculations for three spectral profiles (Lorentzian, Gaussian and rectangular) that Mandel's formula holds with very good accuracy over a wide range of conditions.

The purpose of the present paper is to show that the multimode formulae obtained by Peřina and Horák (1969) (see also Peřina 1970, 1971) for the superposition of coherent and chaotic fields can be used to obtain with good accuracy the integrated intensity distribution, the photon-counting distribution and its factorial moments for arbitrary spectral composition of light in the same way as Mandel's formula has been used for chaotic light. The validity of the approximate formulae is verified by calculating the third factorial moment of the photon-counting distribution according to the exact and approximate formulae for the Lorentzian spectrum. We assume a field produced as the superposition of a one-mode coherent field and a chaotic field with Lorentzian profile of the spectrum.

In § 2 the exact and approximate formulae for the factorial moments are compared and a procedure is given allowing one to obtain the number of degrees of freedom involved in the approximate formulae. In § 3 a discussion of results is given; the use of the approximate formulae in the determination of the spectral parameters is discussed in particular.

2. Comparison of exact and approximate expressions for moments

The following expression can be obtained for the factorial moments $\langle W^k \rangle$

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(Peřina and Horák 1969):

$$\langle W^k \rangle = \sum_{j=0}^{k} \frac{k!}{(k-j)!\Gamma(j+M)} \{\langle n_C\rangle(1-x^2)\}^{k-j} \left(\frac{\langle n_{Ch}\rangle}{M}\right)^j \mathrm{L}_j^{M-1}\left(-\frac{\langle n_C\rangle x^2 M}{\langle n_{Ch}\rangle}\right) \quad (1)$$

where Γ is the gamma function, L_j^M are the Laguerre polynomials, $\langle n_C\rangle$ and $\langle n_{Ch}\rangle$ are the mean photon occupation numbers in the coherent and chaotic fields respectively, M is the number of modes and $x = \sin(\Omega/2)/(\Omega/2)$; here $\Omega = (\omega_C - \omega_0)T$, where ω_C is the frequency of the coherent field, ω_0 is the mean frequency of the chaotic field and T is a time interval of detection. For $k = 2$ and 3 we obtain from (1):

$$\langle W^2 \rangle = \langle n\rangle^2 + \frac{1}{M}(\langle n_{Ch}\rangle^2 + 2x^2\langle n_{Ch}\rangle\langle n_C\rangle) \quad (2)$$

and

$$\langle W^3 \rangle = \langle n\rangle^3 + \frac{3}{M}(\langle n_{Ch}\rangle^3 + \langle n_{Ch}\rangle^2\langle n_C\rangle + 2x^2\langle n_{Ch}\rangle\langle n_C\rangle\langle n\rangle)$$
$$+\frac{2}{M^2}(\langle n_{Ch}\rangle^3 + 3x^2\langle n_{Ch}\rangle^2\langle n_C\rangle) \quad (3)$$

where $\langle n\rangle = \langle n_C\rangle + \langle n_{Ch}\rangle = \langle W\rangle$.

The exact second moment can be obtained by the correlation function technique in the form (Morawitz 1966, Mandel and Wolf 1966)

$$\langle W^2 \rangle = \langle n\rangle^2 + \frac{\langle n_{Ch}\rangle^2}{T^2}\int_0^T\!\!\int |\gamma_{Ch}(t_1-t_2)|^2\,\mathrm{d}t_1\,\mathrm{d}t_2$$
$$+2\frac{\langle n_{Ch}\rangle\langle n_C\rangle}{T^2}\mathrm{Re}\int_0^T\!\!\int \gamma_{Ch}(t_1-t_2)\gamma_C(t_2-t_1)\,\mathrm{d}t_1\,\mathrm{d}t_2 \quad (4)$$

where γ_{Ch} and γ_C are the degrees of coherence for the chaotic and coherent fields, respectively. For the third factorial moment the following exact expression has been obtained (Peřina and Mišta 1968):

$$\langle W^3 \rangle = \langle n\rangle^3 + 3\frac{\langle n\rangle\langle n_{Ch}\rangle^2}{T^2}\int_0^T\!\!\int |\gamma_{Ch}(t_1-t_2)|^2\,\mathrm{d}t_1\,\mathrm{d}t_2$$
$$+2\frac{\langle n_{Ch}\rangle^3}{T^3}\mathrm{Re}\int\!\!\int_0^T\!\!\int \gamma_{Ch}(t_1-t_2)\gamma_{Ch}(t_2-t_3)\gamma_{Ch}(t_3-t_1)\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3$$
$$+6\frac{\langle n\rangle\langle n_{Ch}\rangle\langle n_C\rangle}{T^2}\mathrm{Re}\int_0^T\!\!\int \gamma_{Ch}(t_1-t_2)\gamma_C(t_2-t_1)\,\mathrm{d}t_1\,\mathrm{d}t_2$$
$$+6\frac{\langle n_{Ch}\rangle^2\langle n_C\rangle}{T^3}\mathrm{Re}\int\!\!\int_0^T\!\!\int \gamma_{Ch}(t_1-t_2)\gamma_{Ch}(t_2-t_3)\gamma_C(t_3-t_1)\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3 \quad (5)$$

Considering these exact results for Lorentzian light, so that

$$\gamma_{Ch}(\tau) = \exp\{-(\Gamma|\tau| + \mathrm{i}\omega_0\tau)\} \quad (6)$$

where Γ is the halfwidth of the spectrum, and taking into account that

$$\gamma_C(\tau) = \exp(-\mathrm{i}\omega_C\tau) \quad (7)$$

we obtain for the second factorial moment (Jakeman and Pike 1969)

$$\langle W^2 \rangle = \langle n \rangle^2 \left(1+\frac{1}{N}\right) \tag{8}$$

where

$$N = \langle n \rangle^2 \Bigg\{ \langle n_{\mathrm{Ch}} \rangle^2 \left(\frac{1}{\gamma}+\frac{1}{2\gamma^2}\{\exp(-2\gamma)-1\}\right) + 2\langle n_{\mathrm{Ch}} \rangle \langle n_{\mathrm{C}} \rangle \\ \times \left(\frac{2\gamma}{\Omega^2+\gamma^2}+\frac{2(\Omega^2-\gamma^2)}{(\Omega^2+\gamma^2)^2}+\frac{2\exp(-\gamma)\{(\gamma^2-\Omega^2)\cos\Omega-2\gamma\Omega\sin\Omega\}}{(\Omega^2+\gamma^2)^2}\right)\Bigg\}^{-1} \tag{9}$$

and $\gamma = \Gamma T$; for the third factorial moment, after rather complicated mathematics:

$$\langle W^3 \rangle = \langle n \rangle^3 \left(1+\frac{3}{N}\right) + 3\langle n_{\mathrm{Ch}} \rangle^3 \left(\frac{1}{\gamma^2}\{1+\exp(-2\gamma)\} - \frac{1}{\gamma^3}\{1-\exp(-2\gamma)\}\right) \\ + 6\langle n_{\mathrm{Ch}} \rangle^2 \langle n_{\mathrm{C}} \rangle \Bigg\{ \frac{2}{\Omega^2+\gamma^2} \\ \times \left(1+\frac{2\exp(-\gamma)(\gamma^2\cos\Omega-\gamma\Omega\sin\Omega)-\exp(-2\gamma)(\Omega^2+\gamma^2)+(\Omega^2-\gamma^2)}{2\gamma(\Omega^2+\gamma^2)}\right. \\ + \frac{(\gamma^2-\Omega^2)\{1+\exp(-\gamma)\cos\Omega\} - \Omega\exp(-\gamma)\sin\Omega(2\gamma+1) + \gamma\{\exp(-\gamma)\cos\Omega-1\}}{\Omega^2+\gamma^2} \\ \left. + \frac{2\exp(-\gamma)\{\gamma(\cos\Omega-\exp\gamma)(\gamma^2-3\Omega^2)+\Omega\sin\Omega(\Omega^2-3\gamma^2)\}}{(\Omega^2+\gamma^2)^2}\right)\Bigg\}. \tag{10}$$

In order to use the above-mentioned formulae, derived by Peřina and Horák (1969), for the integrated intensity distribution, the photon-counting distribution and its factorial moments, as approximate formulae for light having the statistical properties of the superposition of coherent and chaotic fields of Lorentzian spectrum, we adjust the parameter M in such a way that the second factorial moment (2) coincides with the exact expression (8). This gives for M:

$$M = N \frac{\langle n_{\mathrm{Ch}} \rangle^2 + 2x^2 \langle n_{\mathrm{Ch}} \rangle \langle n_{\mathrm{C}} \rangle}{\langle n \rangle^2} \tag{11}$$

where N is given by (9).

Another approximation can be obtained by using the corresponding formulae with $x = 1 (\Omega = 0)$ first derived in Peřina (1967, 1968 a,b), for example, for the factorial moments we have

$$\langle W^k \rangle = \frac{k!}{\Gamma(k+M')} \left(\frac{\langle n_{\mathrm{Ch}} \rangle}{M'}\right)^k L_k^{M'-1}\left(-\frac{\langle n_{\mathrm{C}} \rangle M'}{\langle n_{\mathrm{Ch}} \rangle}\right). \tag{12}$$

The parameter M' can be obtained from (11) as

$$M' = N\langle n \rangle^{-2}(\langle n_{\mathrm{Ch}} \rangle^2 + 2\langle n_{\mathrm{Ch}} \rangle \langle n_{\mathrm{C}} \rangle).$$

This approximation has an advantage in such cases where x is not known and cannot be estimated at least approximately.

3. Discussion of results

One can find from (11) that $M \to 1$ for $\gamma \to 0$ (cf. figure 4). This limit provides the corresponding formulae for the superposition of narrow-band chaotic light and one-mode coherent light (Jakeman and Pike 1969, Peřina and Horák 1969).

The reduced second factorial moment $\langle W^2\rangle/\langle W\rangle^2-1$ for $\langle n_C\rangle : \langle n_{Ch}\rangle = 18:2$ ($\langle n_C\rangle+\langle n_{Ch}\rangle = 20$) as a function of γ and Ω is demonstrated by the surface shown in figure 1.

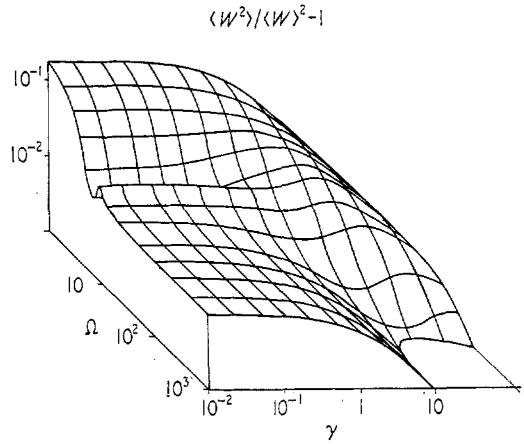

Figure 1. The surface representing the reduced second factorial moment as a function of γ and Ω for $\langle n_C\rangle : \langle n_{Ch}\rangle = 18:2$ calculated on the basis of equation (8).

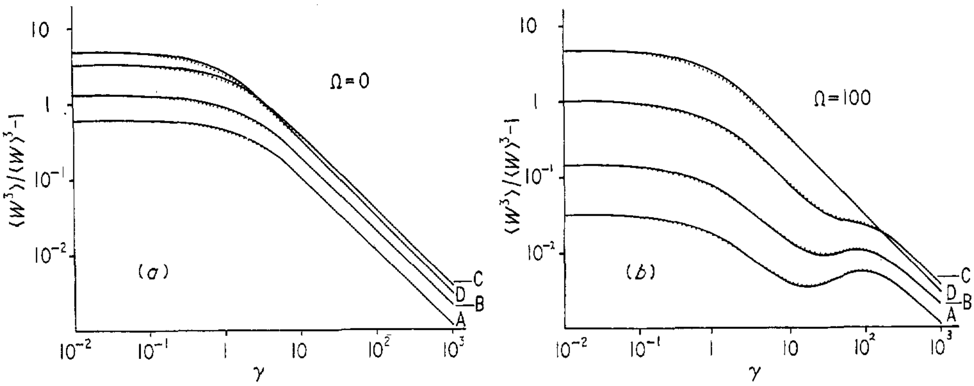

Figure 2. Comparison of the exact values (full curves) and approximate values (dotted curves) for the reduced third factorial moment $\langle W^3\rangle/\langle W\rangle^3-1$ for (*a*) $\Omega = 0$ and (*b*) $\Omega = 100$. The curves A, B, C and D are obtained for $\langle n_C\rangle : \langle n_{Ch}\rangle = 18:2$, $16:4$, $10:10$ and $0:20$ respectively ($\langle n_C\rangle+\langle n_{Ch}\rangle = 20$).

In figure 2 one can see the reduced third factorial moment plotted against γ for various ratios $\langle n_C\rangle : \langle n_{Ch}\rangle$ and for $\Omega = 0$ and 100. The full curves are obtained on the basis of the exact expression (10) while the dotted curves are obtained on the basis of the approximate expression (3) with M given by (11). It may be seen that the agreement is very good. Such a good agreement occurs for all values of the parameters involved. In general the accuracy of approximation increases with increasing $\langle n_C \rangle$. Nearly the same accuracy was obtained for the approximation using the formulae with $x = 1$ and M'. The significance of the approximate formulae lies in the fact that the exact formulae are not obtainable in general in a close form because of mathematical difficulties. The curves in figure 2 represent cut curves of the corresponding surface $\langle W^3 \rangle / \langle W \rangle^3 - 1$ as a function of γ and Ω, which is of similar character to the surface shown in figure 1. The reduced second and third factorial moments as functions of γ for a parameter $R = (\omega_C - \omega_0)/\Gamma = 100$ ($\Omega = R\gamma$) and $\langle n_C \rangle : \langle n_{Ch} \rangle = 18 : 2$ are shown in figure 3. These curves represent cut curves of the

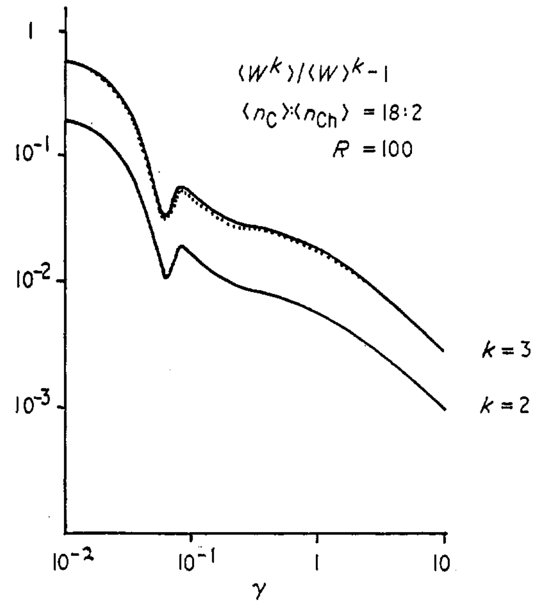

Figure 3. The reduced factorial moments $\langle W^k \rangle / \langle W \rangle^k - 1$ for $k = 2$ and 3 and for $R = 100$ and $\langle n_C \rangle : \langle n_{Ch} \rangle = 18 : 2$. The exact values (full curve) and approximate values (dotted curve) of the reduced third factorial moment are also shown. (The dotted curve corresponds to the M' approximation while the values of the M approximation are practically identical to the full curve.)

corresponding surfaces with the plane $\Omega = 100\gamma$. The curve for the reduced second factorial moment is identical to the corresponding curve in figure 1(a) of the paper by Jakeman and Pike (1969). Note that the curves for $R = 0$ coincide with the curves for $\Omega = 0$.

Good accuracy of the approximate formulae over all regions of the parameters involved provides a way of determining the halfwidth and mean frequency of the spectrum of Gaussian–Lorentzian light. In comparison with the exact method proposed by Jakeman and Pike (1969), who used two measurements of the second factorial moment for two detection times T_1 and T_2, this method is approximate but it provides simply the spectral parameters in a close form. As was pointed out by the referee, error of the third factorial moment in the region $\gamma \simeq 1$ for purely chaotic light is about 5% which would give an error of 15% in the linewidth. So the accuracy of the present method may be less than is suggested by the figures of the third factorial moment, as a consequence of a complicated function dependence. However, the presence of the coherent component leads to an increase of the accuracy in the third factorial moment. For example, for $\Omega = 1$ in the region $\gamma \simeq 1$ we obtain the error practically 0% for $\langle n_C \rangle : \langle n_{Ch} \rangle = 18 : 2$, 0·2% for 16 : 4, 1·3% for 12 : 8, 2·1% for 10 : 10, 5·1% for 4 : 16 and 6·7% for 0 : 20 ($\langle n_C \rangle + \langle n_{Ch} \rangle = 20$). A similar

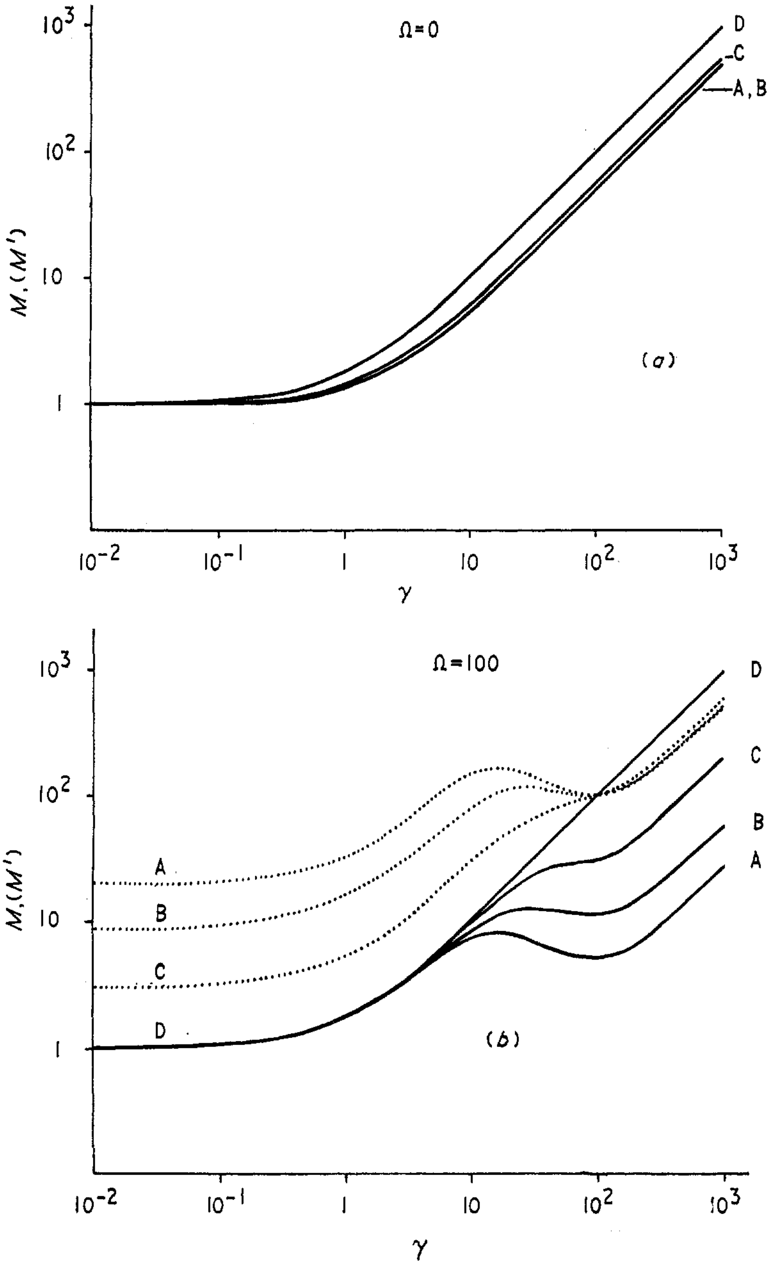

Figure 4. The functions $M(\gamma)$, $M'(\gamma)$ are shown by the full and dotted curves respectively for (a) $\Omega = 0$ and (b) $\Omega = 100$. The curves A, B, C and D correspond to $\langle n_C \rangle : \langle n_{Ch} \rangle = 18 : 2$, $16 : 4$, $10 : 10$ and $0 : 20$ respectively ($\langle n_C \rangle + \langle n_{Ch} \rangle = 20$). In the case $\Omega = 0$ as well as $\langle n_C \rangle = 0$ the functions $M(\gamma)$ and $M'(\gamma)$ are identical.

situation occurs for all other values of Ω. This leads to an increase of the accuracy in determining the spectral parameters. Although it is difficult to give an estimate of this accuracy, which will depend on the position of a point on the surface representing the third factorial moment as a function of γ and Ω, it may be said that the resulting error $\{(\Delta\gamma)^2+(\Delta\Omega)^2\}^{1/2}$, where $\Delta\gamma$ and $\Delta\Omega$ are errors in γ and Ω respectively, will be proportional to the error in the third factorial moment so that these two errors $\Delta\gamma$ and $\Delta\Omega$ will be generally less than the resulting error.

Solving the system of equations (2) and (3) for M and x we arrive at

$$M^{-1} = \frac{3\langle(\Delta W)^2\rangle}{2\langle n_{\mathrm{Ch}}\rangle^2} - \frac{1}{2\langle n_{\mathrm{Ch}}\rangle^2}[9\{\langle(\Delta W)^2\rangle\}^2 - 4\langle(\Delta W)^3\rangle\langle n_{\mathrm{Ch}}\rangle]^{1/2} \tag{13}$$

$$x^2 = \frac{M\langle(\Delta W)^2\rangle - \langle n_{\mathrm{Ch}}\rangle^2}{2\langle n_{\mathrm{Ch}}\rangle\langle n_{\mathrm{C}}\rangle}. \tag{14}$$

From the curve $x = \sin(\Omega/2)/(\Omega/2)$ one can determine Ω, and from the curves given in figure 4—demonstrating dependence of $M(M')$ on γ—one can determine Γ for M obtained from (13). Since ω_{C} and T are fixed, we have the equation $(\omega_{\mathrm{C}}-\omega_0)T = \Omega$ for the mean frequency ω_0. The moments $\langle W^2\rangle$ and $\langle W^3\rangle$ in (13) and (14) may be determined with the help of a photon-counting measurement.

We have limited our considerations to the one-mode coherent field. If more than one coherent frequency is present in the field, generalization is straightforward (cf. Peřina and Horák 1969). Only determination of x may happen to be more complicated. However, having two free parameters M and x in the equations in this case the above given formulae may be used for another approximation in which M and x are adjusted (by means of (13) and (14)) in such a way that the values of the second and the third factorial moments are exact.

Although the present analysis is limited by the assumption of a Lorentzian profile of Gaussian light, the analysis by Bédard *et al.* (1967), performed for Gaussian light of various spectral profiles, justifies our expectation of good accuracy of the approximate formulae for other profiles of the spectrum.

The results of this paper can also be used if for example the parameters $\langle n_{\mathrm{C}}\rangle$ and ω_{C} are modulated in a certain way. This case as well as a generalization to partially polarized light, including multiphoton absorption, will be dealt with in a forthcoming paper.

Acknowledgments

The authors thank Dr Z. Braunerová and Mr T. Kojecky of the Computer Center of Palacký University for their help with calculations.

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